Geometric characterization of driven cofactor systems

2008 J. Phys. A: Math. Theor. 41 042001

(http://iopscience.iop.org/1751-8121/41/4/042001)

IOP FTC ▶▶▶

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 042001 (10pp) doi:10.1088/1751-8113/41/4/042001

FAST TRACK COMMUNICATION

Geometric characterization of driven cofactor systems

W Sarlet and W Vanbiervliet

Department of Mathematical Physics and Astronomy, Ghent University, Krijgslaan 281, B-9000 Ghent, Belgium

Received 12 November 2007, in final form 7 December 2007
Published 15 January 2008
Online at stacks.iop.org/JPhysA/41/042001

Abstract
We discuss the geometry of partially decoupling (submersive) second-order equations in general and illustrate the theory with an application to the case of Lagrangian systems of mechanical type: it is shown that submersiveness implies decoupling into separate subsystems in that case, unless non-conservative forces are added to the system. The main purpose of the paper is to explain the geometric structures underlying so called driven cofactor systems, which constitute a special class of non-conservative Lagrangian systems. In doing so, we generalize the original set-up of driven cofactor systems (Lundmark and Rauch-Wojciechowski 2002 *J. Math. Phys.* **43** 6166) from a Euclidean space to an arbitrary Riemannian one.

PACS numbers: 45.20.Jj, 02.40.Yy
Mathematics Subject Classification: 37J35, 53D17, 53C21, 70H03

1. Introduction

The class of Newtonian equations which is often referred to as cofactor systems nowadays, was the subject of study in Lundmark's thesis [7]. In a first paper about such systems [16], the discussion was mainly restricted to two dimensions and Lundmark's more general account for arbitrary dimension [8] only appeared much later. Specifically, the Newtonian systems which Lundmark first considered were second-order equations (SODEs) of the form:

$$\ddot{q}^i = -(A(q)^{-1})^{ij}\frac{\partial W}{\partial q^j},$$

where A is the cofactor matrix of a matrix G (called planar inertia tensor in [1]) of the form

$$G^{ij}(q) = \alpha q^i q^j + \beta^i q^j + \beta^j q^i + \gamma^{ij}.$$

As observed in [3], such equations can be viewed as representing a class of non-conservative Lagrangian systems

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial q^i} = Q_i,$$

with a 'Euclidean kinetic energy' $T = \frac{1}{2}\sum(\dot{q}^i)^2$ and non-conservative forces of some *quasi-potential* type. The point of this remark is that it paves the way for an intrinsic geometrical description, allowing at the same time for a more general 'Riemannian kinetic energy' $T = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j$. As a matter of fact, within the class of non-conservative Lagrangian systems on a tangent bundle TM, determined by such a T (and hence by some g on M) and by a 1-form $\mu = Q_i(q)\mathrm{d}q^i$ representing non-conservative forces independent of velocities, the geometric definition of a *cofactor system* reads as follows [3].

Definition 1. *The pair* (g, μ) *on a Riemannian manifold* M *determines a cofactor system if* g *admits a special conformal Killing tensor* J *and* μ *satisfies* $D_J\mu = 0$, *where (for any 1-form* ρ*)*

$$D_J\rho = d_J\rho + d(\operatorname{tr} J)\wedge\rho \qquad \textit{or} \qquad D_J\rho = (\det J)^{-1}d_J((\det J)\rho).$$

The term *special conformal Killing tensor* was probably first used in [3] as well, but the same concept or a closely related one appears under different names in different contexts (see, for example, the review paper [2]).

Definition 2. *A special conformal Killing tensor (scKt) w.r.t.* g *is a (non-singular) type* $(1, 1)$ *tensor on* M*, such that* $J_{ij} = g_{ik}J^k_j$ *is symmetric and its covariant derivative (w.r.t. the Levi-Civita connection) satisfies*

$$J_{ij|k} = \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik}),$$

for some 1-form α.

It follows from the scKt condition that $\alpha = d(\operatorname{tr} J)$. More importantly, we have that J has vanishing Nijenhuis torsion:

$$\mathcal{N}_J = 0 \quad\Rightarrow\quad d_J^2 = 0 \quad\Rightarrow\quad D_J^2 = 0,$$

but D_J is not a derivation in the strict sense (it is a so-called gauged differential operator, see [4]). Note that the scKt condition is what produces the matrix G of the Euclidean case above.

The interest of cofactor systems comes from the following further properties. If A is the cofactor tensor of J, i.e. $AJ = (\det J)I$ then A is a Killing tensor. Also, $D_J\mu = 0$ (in view of ${D_J}^2 = 0$) implies that there exists a function W on M, such that $A(\mu) = \mathrm{d}W$ and we have that $E = \frac{1}{2}A_{ij}(q)v^iv^j + W(q)$ is a first integral, which is in fact the Hamiltonian of a quasi-Hamiltonian description of the system. Indeed, denoting by $\widetilde{J}$ the complete lift of J to T^*M, $\widetilde{J}$ commutes with the standard Poisson map $P_0 : \mathcal{X}^*(T^*M) \to \mathcal{X}(T^*M)$, has vanishing Nijenhuis torsion as well and is in fact the recursion operator of a Poisson–Nijenhuis structure on T^*M, with P_0 and $P_J = \widetilde{J}\circ P_0$ as compatible Poisson maps. Then, if $\hat{\Gamma} \in \mathcal{X}(T^*M)$ denotes the image of the given second-order dynamics Γ under the Legendre transform of the kinetic energy Lagrangian $L = \frac{1}{2}g_{ij}v^iv^j$, we have the following important characterization of cofactor systems [3].

Theorem 1. *The cofactor system properties are necessary and sufficient for* $\hat{\Gamma}$ *to have the quasi-Hamiltonian representation:*

$$F\hat{\Gamma} = P_J(dH) \qquad \textit{with} \quad H = \tfrac{1}{2}A^{ij}p_ip_j + W, \quad F = \det J.$$

Cofactor pair systems constitute an interesting subcase: the non-conservative system (g, μ) then has a double cofactor representation, leading to two modified Poisson tensors which are compatible and a hierarchy of first integrals which are in involution with respect to both Poisson structures [3]. For a recent contribution to the subject, see [10].

The motivation for the present paper stems from the study of what we will call *driven cofactor systems*, initiated in [9]. Roughly, these are cofactor systems which exhibit partial decoupling into a 'driving' and a 'driven' subsystem. Explicitly, the equations under consideration in [9] are of the form

$$\begin{aligned} \ddot{y}^i &= Q^i(y^j), & i &= 1,\dots,m \\ \ddot{x}^a &= Q^a(y^i, x^b), & a &= 1,\dots,n \end{aligned}$$

and apart from the overall assumption that the system in (y^i, x^a) is of cofactor type (w.r.t. the Euclidean metric), an extra hypothesis is: after solving the *driving equations* for the y^i, the reduced *driven system*

$$\ddot{x}^a = Q^a(y^i(t), x^b)$$

has a Lagrangian of mechanical type $T - V$, of course with a potential V which will be time-dependent through its dependence on the y-variables. What is proved then are the following quite remarkable features: (i) the driving system is of cofactor type in its own right on the Euclidean space $\mathbb{R}^m$; (ii) for any solution $y(t)$ of the driving system, the driven system has n (time-dependent) integrals; (iii) under some technical assumptions, there exists a canonical transformation, which brings the driven system in a form in which its (time-dependent) Hamilton–Jacobi equation can be solved by separation of variables. The authors provide no clue, however, about the question whether there is any geometrical content for such systems or, expressed differently, whether there exists a coordinate-free characterization of the defining properties of such systems, which then should hold preferably for an arbitrary Riemannian metric again, rather than the Euclidean one.

Our aim is to show that the concept of driven cofactor systems indeed makes sense on an arbitrary Riemannian manifold and can be given a concise coordinate-free description. We will take this opportunity, however, to start from the question of partial decoupling of SODEs in general and then gradually narrow the subject to come to the situation of driven cofactor systems. The point is that, since the work on complete decoupling of SODEs in [14] and [15], we dare claim that the perfect geometrical tools are available for studying partial decoupling as well, but they seem to be hardly known. For this reason, we will focus here on the broad geometrical picture and leave the rather technical computational details of the third of the challenges raised by the results in [9] for a forthcoming paper.

2. Generalities about second-order equations

It is well known that a SODE field on TM

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q, v) \frac{\partial}{\partial v^i}$$

comes with a canonically defined connection on $\tau : TM \to M$, determined by the horizontal lift construction

$$X \in \mathcal{X}(M) \quad \mapsto \quad X^H = \tfrac{1}{2}(X^c + [X^V, \Gamma]).$$

Here, X^c denotes the complete lift and X^V the vertical lift of X. In coordinates, if $X = X^i(q)\partial/\partial q^i$, we have

$$X^H = X^i H_i, \qquad H_i = \frac{\partial}{\partial q^i} - \Gamma_i^j \frac{\partial}{\partial v^j}, \qquad \Gamma_j^i = -\frac{1}{2}\frac{\partial f^i}{\partial v^j}.$$

In turn, with the aid of the projection operators P_H and P_V of this nonlinear connection on TM, one can construct a linear connection on the pullback bundle $\tau^*\tau : \tau^*TM \to TM$

(see, e.g., [11]). Sections of $\tau^*\tau$ are called vector fields along τ and constitute a $C^\infty(TM)$-module $\mathcal{X}(\tau)$. The linear connection $\mathrm{D} : \mathcal{X}(TM) \times \mathcal{X}(\tau) \to \mathcal{X}(\tau)$, said to be of Berwald type, then essentially defines *vertical* and *horizontal covariant derivative operators* D^V_X and D^H_X on $\mathcal{X}(\tau)$. In coordinates, these are determined by the following action on functions $F \in C^\infty(TM)$ and basic vector fields (and then further extend by duality):

$$\mathrm{D}^V_X F = X^i V_i(F), \qquad \mathrm{D}^V_X \frac{\partial}{\partial q^i} = 0 \qquad \left(V_i := \frac{\partial}{\partial v^i}\right)$$

$$\mathrm{D}^H_X F = X^i H_i(F), \qquad \mathrm{D}^H_X \frac{\partial}{\partial q^i} = X^j V_i\big(\Gamma^k_j\big)\frac{\partial}{\partial q^k}.$$

Of equal importance are: the *dynamical covariant derivative* ∇, a self-dual degree 0 derivation on tensor fields along τ, and a (1, 1) tensor Φ along τ, called the *Jacobi endomorphism*. These can be implicitly defined by the following formula for the decomposition of the vector field $\mathcal{L}_\Gamma X^H$ on TM into its horizontal and vertical parts:

$$\mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V.$$

For practical purposes, it suffices to know that:

$$\nabla F = \Gamma(F) \qquad \nabla \frac{\partial}{\partial q^i} = \Gamma^j_i \frac{\partial}{\partial q^j} \qquad \nabla dq^i = -\Gamma^i_j dq^j,$$

$$\Phi^i_j = -\frac{\partial f^i}{\partial q^j} - \Gamma^i_k \Gamma^k_j - \Gamma\big(\Gamma^i_j\big).$$

For the general theory of derivations of forms along τ, one can consult [12, 13].

3. Submersive equations

SODEs for which one can find suitable coordinates in which the equations exhibit partial decoupling are said to be *submersive* by Kossowski and Thompson [6]. These authors used tangent bundle geometry techniques in their analysis and the main relevance of their results is that submersiveness is characterized in an intrinsic way, i.e. by properties which can be verified, in principle, prior to the construction of decoupling coordinates. The point we wish to make is that the tools explained in the preceding section are more concise and ideally suited for studying submersiveness. Indeed, the following result is implicitly present in [14].

Proposition 1. *A SODE Γ on TM is submersive if and only if there exists a distribution K along $\tau : TM \to M$, such that*

$$\Phi(K) \subset K, \qquad \nabla K \subset K, \qquad \mathrm{D}^V_Z K \subset K \;\forall\, Z \in \mathcal{X}(\tau).$$

A sketch of the proof of this statement goes as follows. The D^V-invariance means that K is generated by basic vector fields. D^V- and ∇-invariance further implies D^H-invariance (through the commutator properties of these derivations) and since the horizontal bracket of vector fields along τ reduces to the usual Lie bracket in the case of basic vector fields, the result is that K is generated by a Frobenius integrable distribution. If the x^a are coordinates on integral manifolds of K (and the y^i denote transversal coordinates), ∇-invariance now implies that the forces f^i do not depend on the $\dot{x}^a$. Finally, as can be seen from the coordinate expression of Φ^i_a, Φ-invariance subsequently implies $\partial f^i/\partial x^a = 0$ as well.

It is well known that, if a SODE Γ is Lagrangian, the Hessian of this Lagrangian is a kind of generalized metric on TM, which is invariant under ∇. Let us therefore, for the kind of non-conservative Lagrangian systems we ultimately have in mind, start by investigating what submersiveness means in the presence of a Riemannian metric g on M, satisfying

$$\nabla g = 0 \text{ or equivalently } g_{ij|k} = 0.$$

Denote by $K^{\perp}$ the orthogonal complement of K with respect to g: $g(K, K^{\perp}) = 0$. It easily follows from $\nabla g = 0$ and $\mathrm{D}^V_Z g = 0$ that

$$\nabla K^{\perp} \subset K^{\perp}, \qquad \mathrm{D}^V_Z K^{\perp} \subset K^{\perp}\, \forall\, Z \in \mathcal{X}(\tau).$$

Another general property of Lagrangian systems is that Φ is symmetric with respect to the Hessian of L. Therefore, if our submersive Γ were actually Lagrangian, with the Riemannian g as its Hessian, it would follow that

$$0 = g(\Phi K, K^{\perp}) = g(K, \Phi K^{\perp}),$$

which would imply that also $\Phi(K^{\perp}) \subset K^{\perp}$. Thus we reach the conclusion that a submersive Lagrangian system of mechanical type is necessarily submersive in two different ways. Moreover, the two complementary distributions K and $K^{\perp}$ then are simultaneously integrable, meaning that we obtain the somewhat surprising result that a submersive Lagrangian system (of mechanical type) actually decouples into two separate systems. We will show next that this is not the case in the presence of non-conservative systems, where one can indeed distinguish between a 'driving' and a 'driven' system.

In what follows, we will label the overall coordinates (q, v) of the system with Greek indices (when making use of the kind of general coordinate relations explained so far). Coordinates adapted to the two integrable distributions will typically be represented as $y^i (i = 1, \dots, m)$ and $x^a (a = 1, \dots, n)$, say

$$K = \mathrm{sp}\left\{\frac{\partial}{\partial x^a}\right\}, \qquad K^{\perp} = \mathrm{sp}\left\{\frac{\partial}{\partial y^i}\right\},$$

so that in such coordinates: $(q^\alpha) = (y^i, x^a)(\alpha = 1, \dots, m+n)$.

4. Submersive non-conservative systems

Consider a pair (g, μ) on M, with g Riemannian and μ of the form $\mu = Q_\alpha(q)\,\mathrm{d}q^\alpha$. The corresponding non-conservative SODE then is of the form

$$\Gamma = \widetilde{\Gamma} + Q^\beta \frac{\partial}{\partial v^\beta}, \qquad Q^\beta = g^{\beta\alpha} Q_\alpha,$$

where $\widetilde{\Gamma}$ is the geodesic spray of g.

Comparing the geometrical tools provided by the SODE fields Γ and $\widetilde{\Gamma}$, respectively, we observe first that

$$\Gamma^\alpha_\beta = \widetilde{\Gamma}^\alpha_\beta = \widetilde{\Gamma}^\alpha_{\beta\gamma} v^\gamma.$$

It follows that $H_\alpha = \widetilde{H}_\alpha$, while ∇ and $\widetilde{\nabla}$ coincide on basic tensor fields, but obviously differ on functions on TM. We further have

$$\Phi = \widetilde{\Phi} - \mathrm{D}^{\widetilde{H}} Q, \qquad \text{where} \qquad Q = Q^\alpha \frac{\partial}{\partial q^\alpha}, \quad (\mathrm{D}^{\widetilde{H}} Q)^\alpha_\beta = Q^\alpha{}_{|\beta}.$$

But the geodesic spray $\widetilde{\Gamma}$ is Lagrangian, hence

$$\widetilde{\nabla} g = 0, \qquad \text{and} \qquad \widetilde{\Phi} \lrcorner\, g \text{ is symmetric.}$$

It follows that $\nabla g = 0$ as well, but of course we insist on $\mathrm{d}\mu \neq 0$ to avoid that Γ would also be Lagrangian.

Assume now again that a distribution K makes Γ submersive and consider its complement $K^{\perp}$ with respect to g. As before, since $\nabla g = 0$, we have

$$\nabla K^{\perp} \subset K^{\perp}, \qquad \mathrm{D}^V_Z K^{\perp} \subset K^{\perp}\, \forall\, Z \in \mathcal{X}(\tau),$$

but generally,

$$\begin{aligned} g(\Phi K, K^{\perp}) &= g(\widetilde{\Phi} K, K^{\perp}) - \mathrm{D}^H \mu(K, K^{\perp}) \\ &= g(K, \Phi K^{\perp}) + \mathrm{D}^H \mu(K^{\perp}, K) - \mathrm{D}^H \mu(K, K^{\perp}). \end{aligned}$$

The left-hand side is zero, and to avoid splitting of Γ into two separate subsystems, we want that $g(K, \Phi K^{\perp}) \neq 0$. However, since $\widetilde{\Gamma}$ is Lagrangian and also submersive (and thus splits), we have $g(\widetilde{\Phi} K, K^{\perp}) = 0$ and therefore $\mathrm{D}^H \mu(K, K^{\perp}) = 0$ from the first line. The condition to avoid splitting of Γ thus is that $\mathrm{D}^H \mu(K^{\perp}, K) \neq 0$.

In coordinates (y^i, x^a), simultaneously adapted to K and $K^{\perp}$, the equations of motion will take the form

$$\ddot{y}^i = -\Gamma^i_{jk}(y) \dot{y}^j \dot{y}^k + Q^i(y), \ddot{x}^a = -\Gamma^a_{bc}(x) \dot{x}^b \dot{x}^c + Q^a(y, x).$$

Obviously, these are fairly general conclusions which hold independently of the existence of a special conformal Killing tensor for g.

Within the general class of submersive non-conservative systems, just described, two more assumptions are needed now to capture and at the same time generalize the systems described in [9]. One is the existence of a scKt J which must take care of the overall cofactor nature of the system, the other is simply the assumption $\mathrm{d}\mu(K, K) = 0$, which in adapted coordinates will guarantee that the $Q_a(y, x)$ satisfy

$$\frac{\partial Q_a}{\partial x^b} - \frac{\partial Q_b}{\partial x^a} = 0,$$

and thus that the 'driven' system is (parametrically) Hamiltonian.

Hence we arrive at the following coordinate-free characterization:

Definition 3. *A driven cofactor system is a cofactor system (g, μ, J), determined by a Riemannian metric g, a 1-form μ and a scKt J on M, for which there exists a distribution K along τ, with the properties*

$$\begin{aligned} &\Phi(K) \subset K, \qquad \nabla K \subset K, \qquad \mathrm{D}^V_Z K \subset K, \\ &\mathrm{d}\mu(K, K) = 0, \qquad \mathrm{D}^H \mu(K^{\perp}, K) \neq 0. \end{aligned}$$

5. The cofactor pair nature of driven cofactor systems

We know that in coordinates (y^i, x^a) adapted to the integrable distributions $K^{\perp}$ and K, the metric g will have a block diagonal structure, i.e. $g_{ia} = g_{ai} = 0$. Furthermore, it is easy to verify from $\nabla g = 0$ and the scKt condition for J that

$$\frac{\partial g_{ij}}{\partial x^a} = \frac{\partial J_{ij}}{\partial x^a} = 0, \qquad \frac{\partial g_{ab}}{\partial y^i} = \frac{\partial J_{ab}}{\partial y^i} = 0,$$

and also

$$\begin{aligned} J_{ij|k} &= \tfrac{1}{2}(\alpha_{1i} g_{jk} + \alpha_{1j} g_{ik}), \qquad \alpha_1 = \mathrm{d}\,\mathrm{tr}\, J_1, \qquad J_1 = \left(J^i_j\right), \\ J_{ab|c} &= \tfrac{1}{2}(\alpha_{2a} g_{bc} + \alpha_{2b} g_{ac}), \qquad \alpha_2 = \mathrm{d}\,\mathrm{tr}\, J_2, \qquad J_2 = \left(J^a_b\right). \end{aligned}$$

This means that J_1 and J_2 are in fact special conformal Killing tensors for $g_1 = g|_{K^{\perp}}$ and $g_2 = g|_K$ respectively. But it is worthwhile having a more intrinsic look at the situation.

K and $K^\perp$ being regular, basic, integrable distributions give rise to two complementary sub-bundles of TM (and corresponding sub-modules of $\mathcal{X}(\tau)$). Consider the projection operators

$$P_1 : \mathcal{X}(M) \to K^\perp, \qquad P_2 : \mathcal{X}(M) \to K.$$

Lemma 1. *We have* $\nabla P_i = 0$ *and* $\mathcal{N}_{P_i} = 0$, $i = 1, 2$.

Proof. $\nabla P_i = 0$ easily follows from the ∇-invariance of both distributions and the Nijenhuis property of integrable almost product structures is well known. □

The point about these properties is the following. The projectors P_i obviously have constant trace (think of their representation in adapted coordinates). Therefore, $\nabla P_i = 0$ expresses that the P_i formally satisfy the defining relation of scKts, except that they are degenerate. $\mathcal{N}_{P_i} = 0$ then guarantees that the important Nijenhuis property of scKts remains valid for these degenerate specimens.

We want to show now that P_2 actually can be regarded as providing a second (though degenerate) cofactor representation of the given system. This requires checking that the non-conservative forces μ have the appropriate properties. The following general property of type (1, 1) tensors will be useful for that purpose (we make no notational distinction between the action of a (1, 1) tensor on vector fields or on 1-forms).

Lemma 2. *Let N be a* (1, 1) *tensor field on M. Then the Nijenhuis torsion of N vanishes if and only if for any 1-form* α*:*

$$\mathrm{d}\alpha(NX, NY) + d(N^2\alpha)(X, Y) - d(N\alpha)(NX, Y) - d(N\alpha)(X, NY) = 0.$$

Proof. A fairly straightforward computation shows that the left-hand side is just $-\alpha(\mathcal{N}_N(X, Y))$. □

Lemma 3. *For any* (1, 1) *tensor N with vanishing Nijenhuis torsion and any 1-form* α*, we have*

$$\mathrm{d}\alpha(NX, NY) = d_N(N\alpha)(X, Y).$$

Proof. From the general commutator relation of a type i_* and a type d_* derivation in the standard work of Fröhlicher and Nijenhuis [5], it follows that in the case of a (1, 1) tensor N with vanishing bracket $[N, N]$, we have

$$i_N d_N - d_N i_N = d_{N^2}.$$

Explicitly, this means that for any α,

$$d_N(N\alpha)(X, Y) = d_N\alpha(NX, Y) + d_N\alpha(X, NY) - d_{N^2}\alpha(X, Y),$$

and making use of $d_N = [i_N, d]$ to compute the terms in the right-hand side, we obtain

$$d_N(N\alpha)(X, Y) = 2\mathrm{d}\alpha(NX, NY) + d(N^2\alpha)(X, Y) - d(N\alpha)(NX, Y) - d(N\alpha)(X, NY).$$

The result of the preceding lemma now immediately completes the proof. □

Proposition 2. *Let* (g, μ, J) *be a driven cofactor system. Then the corresponding SODE* Γ *has a second, degenerate cofactor representation with the projector* P_2 *in the role of special conformal Killing tensor.*

Proof. From the preceding lemma, applied to P_2 and the 1-form μ representing the non-conservative forces, we get $\mathrm{d}\mu(P_2X, P_2Y) = d_{P_2}(P_2\mu)(X, Y)$. Therefore, the hypothesis

$\mathrm{d}\mu(K, K) = 0$ becomes equivalent to $d_{P_2}(P_2\mu) = 0$. In addition, thinking in the coordinates (y^i, x^a) adapted to the submersiveness of the system, $P_1\mu$ is a 1-form involving the coordinates y^i of the driving system only, whence the same is true for its exterior derivative. It follows that $i_{P_2}d(P_1\mu) = 0$ and since $P_2(P_1\mu) = 0$ trivially, also $d_{P_2}(P_1\mu) = 0$. Hence, we can conclude that $d_{P_2}\mu = 0$, which is the same as $D_{P_2}\mu = 0$ in view of P_2 having constant trace. □

It follows that we have a kind of degenerate cofactor pair system.

We next proceed to show that the driving system, perhaps not unexpectedly, inherits a cofactor representation in its own right. The type (1, 1) tensors J_1 and J_2, already mentioned at the beginning of this section, constitute two of the following four blocks in which the scKt J can be broken up:

$$J_i = P_i \circ J \circ P_i, i = 1, 2, \qquad J_{12} = P_1 \circ J \circ P_2, \qquad J_{21} = P_2 \circ J \circ P_1.$$

The composition of (1, 1) tensors has to be used with some care here: we think of maps on vector fields in the above defining relations, so the order of the compositions has to be reversed when the dual action on 1-forms is considered. Anyway, we have $J = J_1 + J_2 + J_{12} + J_{21}$, and in the coordinates (y^i, x^a) adapted to the complementary distributions,

$$J_{21} = J_i^a(y, x)\frac{\partial}{\partial x^a} \otimes \mathrm{d}y^i, \qquad J_{12} = J_a^i(y, x)\frac{\partial}{\partial y^i} \otimes \mathrm{d}x^a.$$

Also put

$$\mu_1 = P_1(\mu) = Q_i(y)\,\mathrm{d}y^i, \qquad \mu_2 = P_2(\mu) = Q_a(y, x)\,\mathrm{d}x^a.$$

Lemma 4. $\forall\, X, Y \in K^\perp$, *we have*

$$\mathrm{d}\mu_2(J_{21}X, Y) + \mathrm{d}\mu_2(X, J_{21}Y) - d(J_{21}\mu)(X, Y) = 0.$$

Proof. It is easy to check that, in adapted coordinates, the left-hand side of the above expression reduces to

$$\left(\frac{\partial J_k^a}{\partial y^l} - \frac{\partial J_l^a}{\partial y^k}\right) Q_a X^k Y^l.$$

The general defining relation of a scKt J, when expressed in its type (1, 1) version (cf definition 2 with one index raised), reads

$$J^\alpha_{\beta|\gamma} := \frac{\partial J^\alpha_\beta}{\partial q^\gamma} - J^\alpha_\sigma \Gamma^\sigma_{\beta\gamma} + J^\sigma_\beta \Gamma^\alpha_{\sigma\gamma} = \frac{1}{2}\left(\alpha_\beta \delta^\alpha_\gamma + \alpha_\sigma g^{\sigma\alpha} g_{\beta\gamma}\right).$$

For the present situation, it implies among other things that in adapted coordinates:

$$J^a_{i|k} = \frac{\partial J_i^a}{\partial y^k} - J_j^a \Gamma^j_{ik} = \frac{1}{2}\alpha_b g^{ba} g_{ik}.$$

It follows that

$$\frac{\partial J_i^a}{\partial y^k} = \frac{\partial J_k^a}{\partial y^i},$$

which suffices to arrive at the desired result. □

Proposition 3. *If* (g, μ, J) *determines a driven cofactor system, then the driving system has itself a cofactor representation, determined by* $g_1 = g|_{K^\perp}$, $\mu_1 = P_1(\mu)$ *and* $J_1 = P_1 \circ J \circ P_1$.

Proof. We know that

$$D_J\mu = i_J\,\mathrm{d}\mu - d(J\mu) + d(\mathrm{tr}\, J) \wedge \mu = 0.$$

Applying this in particular to vector fields belonging to $K^{\perp}$ and taking the decomposition of J and μ into account, this reduces to: $\forall\, X, Y \in K^{\perp}$,

$$0 = \mathrm{d}\mu_1(J_1X, Y) + \mathrm{d}\mu_1(X, J_1Y) - d(J_1\mu_1)(X, Y) + d(\operatorname{tr} J_1) \wedge \mu_1(X, Y)$$
$$+ \mathrm{d}\mu_2(J_{21}X, Y) + \mathrm{d}\mu_2(X, J_{21}Y) - d(J_{21}\mu)(X, Y).$$

The second line vanishes in view of the preceding lemma and the remaining terms then express that $D_{J_1}\mu_1 = 0$, which is what we want. □

It follows that the driving system has a quadratic integral which in adapted coordinates reads

$$E = \tfrac{1}{2}A^1_{ij}(y)\dot{y}^i\dot{y}^j + W^1(y),$$

with $A^1 = \operatorname{cof} J_1$ and $A^1\mu_1 = \mathrm{d}W^1$.

Let us finally come back to the cofactor pair nature of the full system. The algorithm for generating a hierarchy of first integrals of a cofactor pair system, as described for example in [3], can be suitably adapted to the case where one of the special conformal Killing tensors is degenerate. In the present situation, if $n = \dim K$ as before, the adapted algorithm can be shown to produce $n + 1$ integrals, one of which is E. Along solutions of the driving system, the other n are time-dependent integrals of the driven system. All of this generalizes the results of [9] for the Euclidean case (cf the points (i) and (ii) mentioned towards the end of the introduction). There is a lot to be said about point (iii) of that enumeration, which involves, among other things, the construction of a specific canonical transformation aimed at obtaining a (time-dependent) Hamiltonian representation of the driven system, which can be solved by separation of variables in the Hamilton–Jacobi equation. In fact, a number of the interesting features of this latter part have remained virtually unnoticed even in the Euclidean situation of [9]. For this reason and because it requires more technicalities which would take us too far away from the geometrical theory of submersive systems we wanted to highlight here, all these additional aspects will be treated in a separate paper.

References

[1] Benenti S 1992 Inertia tensors and Stäckel systems in the Euclidean spaces *Rend. Sem. Mat. Torino* **50** 315–41
[2] Benenti S 2005 Special symmetric two-tensors, equivalent dynamical systems, cofactor and bi-cofactor systems *Acta Appl. Math.* **87** 33–91
[3] Crampin M and Sarlet W 2001 A class of non-conservative Lagrangian8s systems on Riemannian manifolds *J. Math. Phys.* **42** 4313–26
[4] Crampin M, Sarlet W and Thompson G 2000 Bi-differential calculi, bi-Hamiltonian systems and conformal Killing tensors *J. Phys. A: Math. Gen.* **33** 8755–70
[5] Frölicher A and Nijenhuis A 1956 Theory of vector-valued differential forms *Proc. Ned. Acad. Wetensch. Ser.* A **59** 338–59
[6] Kossowski M and Thompson G 1991 Submersive second-order differential equations *Math. Proc. Camb. Phil. Soc.* **110** 207–24
[7] Lundmark H 2001 Newton systems of cofactor type in Euclidean and Riemannian Spaces *PhD Thesis* Linköping University, Sweden
[8] Lundmark H 2003 Higher-dimensional integrable Newton systems with quadratic integrals of motion *Stud. Appl. Math.* **110** 257–96
[9] Lundmark H and Rauch-Wojciechowski S 2002 Driven Newton equations and separable time-dependent potentials *J. Math. Phys.* **43** 6166–94
[10] Marciniak K and Blaszak M 2007 Non-Hamiltonian systems separable by Hamilton–Jacobi method *Preprint* 0707.1113v1
[11] Martínez E and Cariñena J F 1996 Geometric characterization of linearizable second-order differential equations *Math. Proc. Camb. Phil. Soc.* **119** 373–81

[12] Martínez E, Cariñena J F and Sarlet W 1992 Derivations of differential forms along the tangent bundle projection *Differ. Geom. Appl.* **2** 17–43

[13] Martínez E, Cariñena J F and Sarlet W 1993 Derivations of differential forms along the tangent bundle projection II *Differ. Geometry Appl.* **3** 1–29

[14] Martínez E, Cariñena J F and Sarlet W 1993 Geometric characterization of separable second-order equations *Math. Proc. Camb. Phil. Soc.* **113** 205–24

[15] Sarlet W and Thompson G 2001 Complex second-order differential equations and separability *Appl. Algebra Eng. Commun. Comp.* **11** 333–57

[16] Rauch-Wojciechowski S, Marciniak K and Lundmark H 1999 Quasi-Lagrangian systems of Newton equations *J. Math. Phys.* **40** 6366–98